Structural Modification of Nonlinear FEA Subcomponents Using Nonlinear Normal Modes

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ABSTRACT

Structural modification procedures are widely used to predict changes in the dynamics of a structure based on the addition of stiffeners, reinforcements, modifications of bolted joints, and payloads. A truncated modal basis of a linear dynamic system can be used to predict the changes of the mode shapes and frequencies due these modifications. This work proposes an extension of modal structural modification for geometrically nonlinear structures, by representing the structure with a set of nonlinear normal modes. An approximate quasi-linear modal model is defined from the fundamental frequency and maximum deformation shape of the nonlinear normal mode solutions. The resulting quasi-linear model has energy dependent mode shapes and natural frequencies. An iterative algorithm then applies modal structural modification to the quasi-linear modal model such that the modal parameters used in the substructuring routine are appropriate for each response level of interest. The method is demonstrated on a finite element model of a geometrically nonlinear beam with a variable elastic boundary condition. The model is meant to mimic the uncertain boundary conditions of a substructure in a hypersonic air vehicle. The nonlinear normal modes are computed for the unmodified beam, and a torsion spring is added to the boundary using the proposed method. It is found to give accurate predictions of the nonlinear modes of the assembly, potentially at a greatly reduced computational cost.

Keywords: Nonlinear normal modes, structural modification, substructuring, reduced order modeling, nonlinear dynamics.

1. Introduction

Substructuring approaches in structural dynamics have been widely used to predict the changes in dynamics caused by the addition of a structural element, or even after coupling another substructure. Substructuring methods can be classified as to whether the system is represented in either the physical, modal or frequency domain. An example of physical domain coupling is the finite element method, where simple geometric elements are coupled to predict the behavior of an assembly [1]. The geometry of the assembly can be incredibly complicated if enough elements are coupled together, allowing for realistic structures to be modeled. Modal substructuring methods allow one to reduce the model order at the subcomponent level. A truncated modal basis can provide an accurate reduced order model of the system, allowing for an assembly of subcomponents to have significantly fewer degrees-of-freedom (DOF) than would be required for an accurate finite element model. Linear normal modes have been used to couple linear systems for over 50 years, one of the most popular techniques being the Craig-Bampton method [2]. In this work, a modal structural modification technique is extended to nonlinear subcomponent models represented by a truncated set of nonlinear normal modes (NNM). An iterative algorithm is developed in order to quickly estimate the NNMs of a nonlinear structure after the addition of a lumped mass or spring element.

Nonlinearities in structural models are introduced by a variety of phenomena such as large deformations, nonlinear material constitutive laws, buckling, and friction in joints. These physics must be accounted for to accurately predict the behavior of a structure, especially when the linearity assumption is no longer valid. For example, large deformation analysis of a thin, flat, planar structure (such as the stiffened skin panel of a hypersonic air vehicle) shows that the coupling of the axial and bending motions actually reduces the maximal amplitude of the response. Using linear analysis for such predictions would actually over estimate the response amplitude and the dynamic stress, resulting in an overly conservative design. On the other hand, other nonlinear phenomena, such as localization and internal resonance, can drive a structure to far larger stresses than would be predicted by linear theory. In situations such as these, it is important to accurately model nonlinearity to avoid damaging response levels under certain operating conditions. Unfortunately, time simulations of large scale nonlinear finite element models are very computationally expensive and tend to drive design engineers away from nonlinear analysis, if possible. The nonlinear normal mode provides an interpretation of the nonlinear behavior of a structure that can be used as a design tool for models that exhibit nonlinearity, without the need for repeated transient analysis simulations. It is with this motivation that modal analysis techniques are extended to nonlinear systems using NNMs as a basis.

A nonlinear normal mode has been defined as a "not necessarily synchronous periodic response to the conservative nonlinear equations of motion", as developed by Vakakis, Kerschen and others [3, 4]. This definition is an extension of the pioneering work of Rosenberg [5]. The NNM describes how the resonant frequency and deformation shape change with energy. Many key nonlinear phenomena can be described with this approach, such as internal resonance, localization and frequency-energy dependence. In many structural dynamic systems, an NNM can be thought of as an extension of a linear mode shape and frequency at low energy levels. One numerical method for computing NNMs is the shooting and pseudo-arclength continuation technique developed by Peeters et al [6]. Other numerical methods have been developed as well [7-10]. The nonlinear normal mode describes the resonant conditions of a nonlinear structure, as they form the backbone to the nonlinear forced response curves. Although the property of superposition and orthogonality do not apply, NNMs still provide tremendous insight into the behavior of the system.

Several prior works have been put forth to extend substructuring techniques to nonlinear systems in the frequency and modal domain. A substructuring approach was developed by Cömert and Özgüven in [11], where two linear systems in the modal or frequency domain were coupled by a nonlinear spring element. The nonlinear spring element was approximated by a describing function, satisfying a single frequency harmonic balance model. The nonlinear assembly resulted in a nonlinear receptance matrix, which is used to predict the monoharmonic, steady state forced response. In later publications, the method was extended to the attachment of a nonlinear element to a linear system [12] using the frequency based substructuring (FBS) approach presented in [13]. The method was also demonstrated by coupling a linear system to a localized nonlinear structure, represented by a nonlinearity matrix derived from the describing function method [14]. Other frequency domain methods were presented in [15-17], coupling a localized nonlinear element represented by a describing function to the linear FRF of a structural model.

Other substructuring techniques have been developed for nonlinear systems based on the concept of the nonlinear normal mode. Chong & Imregun [18] developed a modal substructuring approach for structures with global nonlinearities. They defined a nonlinear mode as either a parametric modal model fit to nonlinear frequency response measurements, or as the eigenvalues and eigenvectors of the mass and tangent stiffness matrix about a given static configuration. They used an iterative algorithm based on the subcomponent nonlinear modes to satisfy a balance between the modal amplitudes in the assembly and subcomponent models. Another approach developed by Apiwattanalunggarn et al [19] was based on an NNM defined as a two-dimensional invariant manifold in phase space. The nonlinear subcomponent was represented by a single fixed interface NNM and a set of linear constraint modes. The method worked well for a SDOF subcomponent, but the manifolds required to describe higher order nonlinear models are very costly to compute so the method was deemed to be impractical. Recently, Allen & Kuether [20] have proposed a substructuring approach based on the NNM definition of Rosenberg, Vakakis and Kerschen mentioned earlier. The deformation shapes and fundamental frequencies from the exact NNM solutions were used to define a quasi-linear modal model, in which the mode shapes and natural frequencies were dependent on energy. The method of Lagrange multipliers was used to couple two nonlinear structures represented by the quasi-linear modal model. The full set of NNMs were used in this previous publication. The nonlinear structural modification procedure presented here is an extension of this work.

This work presents a nonlinear structural modification procedure that can quickly and accurately predict how the NNMs of the system will change due to the addition of a lumped mass or spring element. This work is motivated by the need to predict the response of realistic structures that are modeled in commercial finite element packages.

The nonlinear modification procedure uses a truncated set of NNMs computed from a nonlinear finite element model, allowing for order reduction at the subcomponent level. The NNMs of the unmodified structure are computed using the algorithm in [6], which relies on numerical time integration and the computation of the Jacobian matrix of the equations of motion. Recently, the authors found that one can accurately predict the NNMs of a structure by combining this algorithm with a nonlinear reduced order model (NLROM) of the structure [7]. This work uses a truncated set of NNMs computed using this approach, to predict how those nonlinear modes would change due to a modification; this avoids the need to re-run the finite element model to numerically recompute the reduced order model and then the nonlinear modes. The proposed method uses the deformations and fundamental frequencies from the exact NNM solution to define a quasi-linear modal model. This model is treated as a linear modal model, although while recognizing that the quasi-linear natural frequencies and deformation shapes vary with energy. An iterative approach is then used with the method of Lagrange multipliers to adjust the modes so they reflect the properties of the structure at the energy level of interest. The result is an estimate of the nonlinear modes of the structure at a certain energy level, and the procedure is easily repeated over the energy range of interest.

The remainder of this paper is outlined as follows. Section 2.1 discusses how the nonlinear normal modes of a geometrically nonlinear finite element model are found using nonlinear reduced order models. A description of the quasi-linear modal model used to approximate the nonlinear system is defined in Section 2.2, and the iterative nonlinear structural modification algorithm is explained in detail in Section 2.3. Section 3 demonstrates the proposed algorithm by modifying the boundary stiffness of a geometrically nonlinear beam. A discussion of the accuracy and limitations of the results are accompanied with a comparison between the modified and exact NNMs.

2. Theoretical Development

2.1 Representing Nonlinear Subcomponents with Nonlinear Normal Modes

A nonlinear normal mode describes the periodic free response of the autonomous, conservative nonlinear equations of motion for a system. In general, the discretized equations for an *N*-DOF system can be written as

$$M\ddot{x} + Kx + f_{NL}(x) = \{0\} \tag{1}$$

where ***M*** is the $N \times N$ mass matrix, ***K*** is the $N \times N$ linear stiffness matrix, and $f_{NL}(x)$ is the $N \times 1$ nonlinear restoring force vector. The displacement, velocity and acceleration are represented by the $N \times 1$ vectors x, $\dot{x}$, and $\ddot{x}$, respectively. The NNMs computed from the full order equations of motion exactly satisfy the periodicity condition, allowing for strong nonlinearities to be captured. The exact multi-harmonic NNM response can be written for the system in Eqn. (1) as a complex Fourier Series of the form

$$x_{NNM}(t) = \sum_{k=-\infty}^{\infty} X_{NNM,k} e^{ik\omega_{NNM}t} \tag{2}$$

where $X_{NNM,k}$ is the complex amplitude of the k^{th} harmonic, ω_{NNM} is the fundamental frequency, and *t* is time. The energy dependence of the solution comes from the fact that the fundamental frequency and complex amplitudes evolve as the amplitude of the response increases. At low energy, the periodic solution branches are in the neighborhood of the linear normal mode solutions of the linearized system. For an *N*-DOF system, there exists N nonlinear normal modes that are nonlinear extensions of linear modes.

In general, the closed form equations of motion are not defined explicitly for the nonlinear finite element models of interest, so a numerical approach is required to compute the NNMs. One recently developed numerical approach is based on a shooting and pseudo-arclength continuation technique, and works by integrating the conservative nonlinear equations of motion and iteratively adjusting the initial conditions until a periodic response is found [6]. While this method is very accurate and effective, it is too computationally expensive to be practical for high fidelity finite element models, especially since the closed form equation of motion is not known. The numerical shooting and continuation techniques rely on the computation of the Jacobian matrix of the equations of motion, and must be computed using finite difference schemes. The Jacobian matrix requires 2N finite difference computations, and may need to be computed several times for a single solution on the NNM branch.

In order to overcome these difficulties, in [7] the authors proposed to use a nonlinear reduced order model to decrease the size of the nonlinear equations of motion, and therefore the size of the Jacobian matrix required by the continuation algorithm. This NLROM can be used with the algorithm in [6] then to calculate the NNMs of the structure. A geometrically nonlinear finite element model with *N*-DOF can be reduced to a low-order system of nonlinear modal equations, based on a linear modal coordinate transformation. The low frequency linear modes are computed from the linear (or linearized) mass and stiffness matrices, ***M*** and ***K***, from the equations of motion in Eqn. (1). A truncated modal basis, shown in Eqn. (3), is used to transform the full order system from *N* physical DOF to a reduced set of *m* generalized modal coordinates. In general, the low frequency modes in the bandwidth of interest are selected for the coordinate transformation, although higher frequency modes may be included.

$$x = [\Phi]q \tag{3}$$

Φ is the $N \times m$ mass normalized mode matrix with each column being a linear mode shape, and q is the $m \times 1$ modal coordinate vector. Applying the coordinate transformation to the full nonlinear system of equations in Eqn. (1), and premultiplying by Φ^T, a nonlinear modal equation can be written as

$$\ddot{q}_r + \omega_r^2 q_r + \theta_r(q_1, q_2, ..., q_m) = 0 \tag{4}$$

where $\theta_r(q_1, q_2, \ldots, q_m)$ is approximated as a polynomial form as

$$\theta_r(q_1, q_2, ..., q_m) = \sum_{i=1}^{m}\sum_{j=i}^{m} B_r(i,j) q_i q_j + \sum_{i=1}^{m}\sum_{j=i}^{m}\sum_{k=j}^{m} A_r(i,j,k) q_i q_j q_k \tag{5}$$

The nonlinear coefficients A_r and B_r are unknown for each modal equation. The resulting equations are nonlinear and fully coupled, but the order is much less than that of the original model ($m << N$). The nonlinearity is approximated by a quadratic and cubic polynomial function of the modal amplitude, and captures stiffening and softening behavior caused by geometric nonlinearities. Several methods exist for determining these nonlinear coefficients from a nonlinear finite element model, as reviewed in [21]. The method chosen for this work is the Implicit Condensation and Expansion (ICE) technique [22]. The ICE method determines the nonlinear coefficients A_r and B_r from a set of nonlinear static solutions to a series of applied loads. The resulting static deformations are used to form a least squares problem that can be solved to fit the nonlinear coefficients in Eqn. (5). For flat structures such as planar beams and plates, an expansion of the membrane displacements caused by the bending displacements is needed to account for axial-bending coupling. The ICE method uses the reduced modal basis and static solutions to compute a set of membrane modes that are orthogonal to the bending modes. The details of the approach are found in [23]. Once the coefficients A_r and B_r are determined, the nonlinear equations (4) and (5) used with the continuation algorithm in [6] in order to compute the NNMs of the structure.

2.2 Quasi-Linear Modal Models

The proposed structural modification method is closely related to the nonlinear substructuring work of Özgüven and his collaborators [11, 12, 14]. In their works the nonlinear subcomponent model is approximated by a fundamental frequency harmonic balance model. Consider the nonlinear equations of motion of the form in Eqn. (1). It is often the case that a single frequency harmonic force causes a response that is well approximated by a single harmonic at the same forcing frequency. The steady state harmonic response can then be written as

$$x = \mathrm{Re}(Xe^{i\omega t}) \tag{6}$$

where X is the complex amplitude of the response and ω is the fundamental response frequency. Under this assumption, the nonlinear restoring force vector is also assumed to be harmonic and can be written as

$$f_{NL}(x) = \mathrm{Re}(F_{NL} e^{i\omega t}) \tag{7}$$

Budak and Özgüven [24] have shown that the harmonic nonlinear restoring force amplitude vector, F_{NL}, can be written as a function of the response amplitude, X, as

$$F_{NL} = \left[K_{NL}(X)\right]X \tag{8}$$

where $\boldsymbol{K_{NL}(X)}$ is an $N \times N$ nonlinear stiffness matrix that is a function of the unknown complex amplitudes, X. The Describing Function Method can be used to generate the nonlinear stiffness matrix for various types of nonlinearities, as described in [25]. Combining Eqns. (6), (7) and (8) with Eqn. (1), the fundamental harmonic balance model becomes

$$\left[-\omega^2 M + K + K_{NL}(X)\right]X = F \tag{9}$$

where F is the complex amplitude vector of the externally applied force. The nonlinear algebraic equations in Eqn. (9) can be iteratively solved for the steady state response X to a harmonic external load F.

This fundamental harmonic approximation to the nonlinear equations of motion can be used with a variety of substructuring techniques, either in the physical or frequency domain. Here, the modification procedure for the addition of a discrete, linear spring element is shown to demonstrate the use of the nonlinear harmonic balance model. The method of Lagrange multipliers is used to couple the two systems; the details of the approach are outlined in [26]. The equation of motion for a discrete spring element attached to point "c" is expressed as

$$k_{spring} x_c = f_{spring} \tag{10}$$

The unconstrained equations of motion for the two uncoupled systems becomes

$$\left[\begin{bmatrix} \left[-\omega^2 M\right] & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} [K] & 0 \\ 0 & k_{spring} \end{bmatrix} + \begin{bmatrix} [K_{NL}(X)] & 0 \\ 0 & 0 \end{bmatrix}\right]\begin{Bmatrix} \{X\} \\ x_c \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ f_{spring} \end{Bmatrix} \tag{11}$$

In order to assemble the two models, force equilibrium and compatibility must be satisfied. The compatibility constraints for the assembly are expressed as

$$[a]\begin{Bmatrix} \{X\} \\ x_c \end{Bmatrix} = 0 \tag{12}$$

The $1\times(N+1)$ vector a consists of an entry of 1 for the connection point on the unmodified structure, and -1 at the last entry related to x_c. The remaining entries are zeros. Using the compatibility condition, the constrained and unconstrained coordinates are related by

$$\begin{Bmatrix} \{X\} \\ x_c \end{Bmatrix} = [B]\{X_u\} \qquad where \qquad B = null(a) \tag{13}$$

Inserting Eqn. (13) into (11), and premultiplying by $\boldsymbol{B}^{\mathrm{T}}$, the unconstrained equations of motion for the modified system becomes

$$[B]^{\mathrm{T}}\left[\begin{bmatrix} \left[-\omega^2 M\right] & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} [K] & 0 \\ 0 & k_{spring} \end{bmatrix} + \begin{bmatrix} [K_{NL}(X)] & 0 \\ 0 & 0 \end{bmatrix}\right][B]\{X_u\} = [B]^{\mathrm{T}}\begin{Bmatrix} \{F\} \\ f_{spring} \end{Bmatrix} \tag{14}$$

This constitutes a set of nonlinear algebraic equations that must be solved iteratively to obtain the monoharmonic, steady state forced response of the modified nonlinear system. Note that a similar procedure can be applied to add a lumped mass or a nonlinear spring element, or it may even be used to couple two nonlinear systems together.

One should note that this procedure presumes that the equations of motion are known explicitly, or at least that the harmonic balance model is known. For the nonlinear finite element models that are of interest in this work, the harmonic balance model $\boldsymbol{K_{NL}(X)}$ is not easy to form, and the system matrices $\boldsymbol{M}$, $\boldsymbol{K}$ and $\boldsymbol{K_{NL}(X)}$ may be prohibitively large. Furthermore, this approach does not provide a mechanism for order reduction at the subcomponent level; each nonlinear subcomponent is represented by a physical model and equilibrium must be satisfied for every node in the assembly.

On the other hand, it is interesting to note that the harmonic balance model has the same form as a linear model, only it is tuned to a specific amplitude of response. This prompts the question: Can an amplitude dependent linear modal model be used to predict the amplitude dependent frequency and deformation shape of a nonlinear structure in a similar way? If this were the case, then perhaps the number of subcomponent modes could also be truncated to reduce the number of generalized coordinates needed to represent the system, reducing the computational cost further. That is the impetus for the methodology proposed in this work. Although a proof for the proposed procedure has not yet been developed, the idea behind the method is intuitive and quite similar to what was outlined above.

Specifically, we propose to represent the substructure of interest using a quasi-linear (QL) modal form that is based on its nonlinear normal modes. This is similar, in spirit at least, to the amplitude dependent harmonic balance model in Eqn. (9), only in the proposed the energy level of each NNM is used as the adjustable parameter rather than the response amplitude of each node. Hence, the quasi-linear modal parameters can be written as

$$\Phi^{QL}(E) = \left[\phi_1^{QL}(E_1) \quad \phi_2^{QL}(E_2) \quad \ldots \quad \phi_P^{QL}(E_P)\right] \tag{15}$$

$$\omega^{QL}(E) = \begin{bmatrix} \omega_1^{QL}(E_1) & 0 & 0 & 0 \\ 0 & \omega_2^{QL}(E_2) & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & \omega_P^{QL}(E_P) \end{bmatrix} \tag{16}$$

where P is the number of NNMs used in the QL model, $\phi_r^{QL}(E_r)$ is the mass normalized QL mode shape of the r^{th} mode, and $\omega_r^{QL}(E_r)$ is the fundamental frequency of the r^{th} mode. The QL mode shape is defined from the initial physical displacement (having a strain energy E_r) that initiates the NNM response of the nonlinear equations of motion. This can be thought of as the maximum displacement field in the NNM response. For a given solution along the NNM curve, the QL mode shape is defined as

$$\phi_r^{QL}(E_r) = \frac{x_{NNM,r}(E_r)}{\sqrt{x_{NNM,r}^{\mathrm{T}}(E_r)[M]x_{NNM,r}(E_r)}} \tag{17}$$

where E_r is the strain energy of the initial physical displacement $x_{NNM,r}(E_r)$. There are many discrete energy solutions for each NNM, each having a different QL mode shape and fundamental frequency. With the QL modal model, modal substructuring approaches (e.g. method of Lagrange multipliers) are used based on the modal equations of motion of the form

$$[I]\ddot{q}^{QL} + \left[\omega^{QL}(E)\right]q^{QL} = 0 \qquad where \qquad x(t) = \left[\Phi^{QL}(E)\right]q_r^{QL} \tag{18}$$

An iterative algorithm is proposed in the following subsection that seeks to satisfy the energy dependence in the QL modes.

2.3 Structural Modification Algorithm using NNMs

The quasi-linear modal model defined in the previous subsection approximates a nonlinear model as a linear system with mode shapes and frequencies that depend on modal energy. The modes are then assembled using the method of Lagrange multipliers, exactly as would be done when assembling linear substructures based on their modal models. This paper focuses on structural modification, so the algorithm is specialized to that case to a certain extent, as detailed below.

1. Compute truncated set of NNMs:

Compute the NNMs of the nonlinear equations of motion for the number of modes desired.

2. Define QL modal model at a fixed energy level:

Take the displacements that initiate the NNM solutions, $x_{NNM,r}(E_r)$, in the truncated basis set at the same energy level E_j. From each displacement $x_{NNM,r}(E_r)$, define the mass normalized quasi-linear mode shape $\phi_r^{QL}(E_r)$ as shown in Eqn. (17). The modal amplitude for the r^{th} mode is then defined as

$$q_r^{QL} = \sqrt{x_{NNM,r}^{\mathrm{T}}(E_r)[M]x_{NNM,r}(E_r)} \tag{19}$$

such that the relation between the physical and modal response is $x_{NNM,r}(E_r) = \phi_r^{QL}(E_r) q_r^{QL}$. Next, specify the natural frequency $\omega_r^{QL}(E_r)$ of each QL mode from the NNM fundamental frequency at E_j, and assemble the QL modal model in Eqn. (18).

3. Apply structural modification procedure to QL modal model:

With the QL modal model, the method of Lagrange multipliers can now be used to couple either a linear spring or lumped mass to the system. For example, attaching a linear spring to the x_c DOF results a set of differential-algebraic equations given by

$$\begin{bmatrix} [I] & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q}^{QL} \\ \ddot{x}_c \end{Bmatrix} + \begin{bmatrix} [\omega^{QL}(E)] & 0 \\ 0 & k \end{bmatrix} \begin{Bmatrix} q^{QL} \\ x_c \end{Bmatrix} = \begin{Bmatrix} [\Phi^{QL}(E)]^T \{F\} \\ F_c \end{Bmatrix} \tag{20}$$

$$[a]\begin{Bmatrix} q^{QL} \\ x_c \end{Bmatrix} = 0 \tag{21}$$

A number of methods exist for solving this set of equations. For structural modification problems, the constrained and unconstrained generalized coordinates can be related as

$$\begin{Bmatrix} q^{QL} \\ x_c \end{Bmatrix} = \begin{bmatrix} I \\ \Phi_c^{QL}(E) \end{bmatrix} \{q^{QL}\} = [B]\{q_u^{QL}\} \tag{22}$$

where q_u^{QL} are the unconstrained coordinates of Eqn. (20). Using the $\boldsymbol{B}$ matrix, the new unconstrained equations of motion satisfying compatibility and force equilibrium become

$$[B]^{\mathrm{T}} \begin{bmatrix} [I] & 0 \\ 0 & 0 \end{bmatrix} [B]\{\ddot{q}_u^{QL}\} + [B]^{\mathrm{T}} \begin{bmatrix} [\omega^{QL}(E)] & 0 \\ 0 & k \end{bmatrix} [B]\{q_u^{QL}\} = [B]^{\mathrm{T}} \begin{Bmatrix} [\Phi^{QL}(E)]^{\mathrm{T}} \{F\} \\ F_c \end{Bmatrix} \tag{23}$$

Next, the eigenvalues and eigenvectors are computed from the equations of motion in Eqn. (23). Resulting from this will be a set of mode shapes Φ^{mod} and frequencies ω^{mod} of the modified QL modal model.

4. Compute the energy distribution among the modes of the unmodified structure:

The generalized coordinates of the unconstrained, modified equations can be related to the generalized coordinates of the unmodified coordinates as

$$\begin{Bmatrix} q^{QL} \\ x_c \end{Bmatrix} = [B]\{q_u^{QL}\} \tag{24}$$

The generalized coordinates for the modified QL mode shapes are the same as the generalized coordinates of the unmodified QL modes, based on the form of the ***B*** matrix in Eqn. (22). Therefore, the physical deformation of each modal response of the modified structure can be written as

$$x_r^{\mathrm{mod}} = \phi_r^{\mathrm{mod}} q_r^{QL} \tag{25}$$

Using each physical deformation x_r^{mod}, the total strain energy in the unmodified structure is computed. The energy distribution of each physical response in Eqn. (25) is defined as

$$E^{\mathrm{mod}} = \begin{bmatrix} E_1^{\mathrm{mod}} & E_2^{\mathrm{mod}} & \cdots & E_P^{\mathrm{mod}} \end{bmatrix} \tag{26}$$

5. <u>Check whether the strain energy of the modified deformation matches the energy in the unmodified modal model:</u>

Define a tolerance condition as

$$\frac{\left\| E - E^{\mathrm{mod}} \right\|_2}{\left\| E^{\mathrm{mod}} \right\|_2} < \varepsilon \tag{27}$$

Ideally, the difference between the energy used in the QL model and the energy of the modified deformation should be equal to zero. Since this is a numerical approach, the algorithm proceeds until the difference in energy is below some tolerance, ε.

5a. If the tolerance is satisfied, then store the modified modal parameters. The approximate NNM of the modified structure for the r^{th} mode is then given by $\phi_r^{QL,\mathrm{mod}}(E_r^{\mathrm{mod}})$, $\omega_r^{QL,\mathrm{mod}}(E_r^{\mathrm{mod}})$ and q_r^{QL}. Increase the energy level that was used to initiate the algorithm and repeat steps 2 through 5.

5b. If the tolerance is not satisfied, then return to step 2 and update the QL modal model parameters $\phi_r^{NNM}(E_r)$ and $\omega_r^{NNM}(E_r)$ at the discrete modal energies E^{mod}. Note that for each iteration, the modal amplitudes q_r^{QL} are not updated. In general, the distribution of energy between each mode will not be the same. Repeat steps 2-5 until the tolerance in Eqn. (27) is satisfied.

Based on the observed performance, the iterative algorithm converges to a balanced solution within 3-8 iterations for an ε value of 1e-4. A schematic of the algorithm is shown below in Fig. 1.

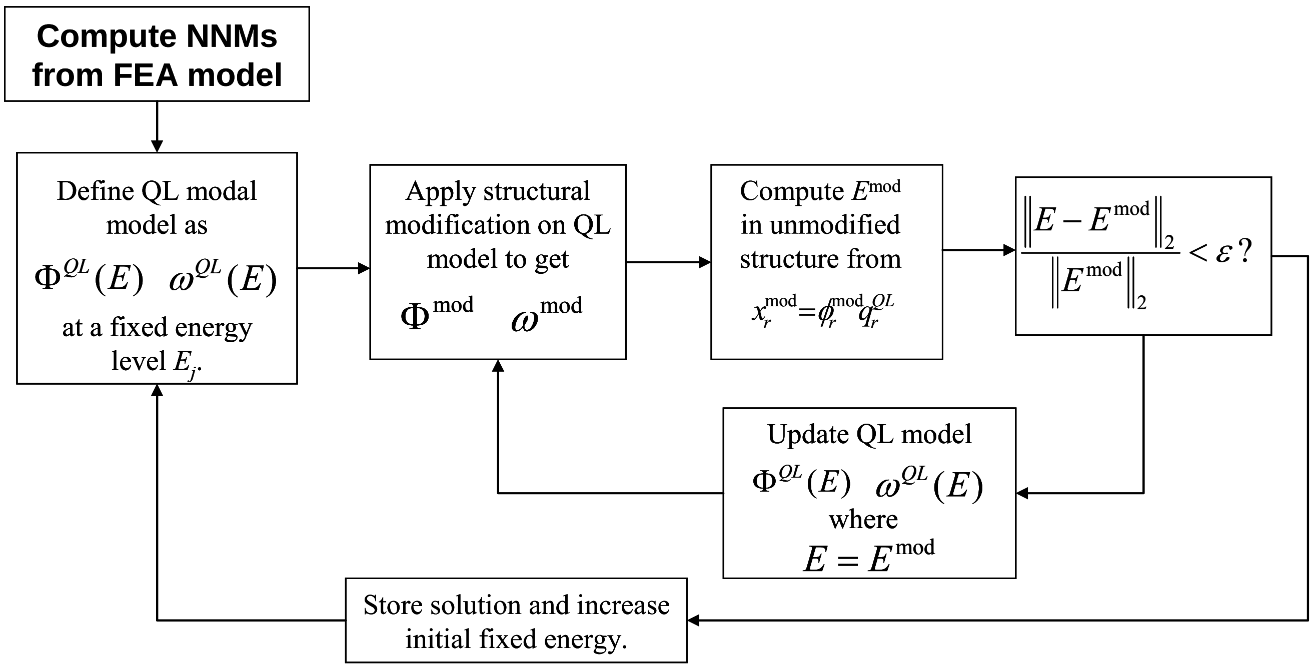

Figure 1: Schematic of structural modification algorithm to predict NNMs of a modified structure.

3. Numerical Results

3.1 Structural Modification of Geometrically Nonlinear Beam

The nonlinear structural modification technique is now applied to a planar, geometrically nonlinear beam that is modeled in Abaqus finite element software. This problem was motivated by prior studies [27, 28] which showed that the nonlinear dynamic response of an inlet ramp panel is sensitive to the stiffness of the structure which it is attached to. A torsion spring is coupled one end, as shown in Fig. 2, to represent the uncertain elastic boundary condition. The model is used to understand the effects of the boundary stiffness on the nonlinear normal modes of the structure. The beam is known to exhibit stiffening nonlinearities due to the coupling between the transverse and axial motions for large deformations. The beam under study is 9 inches in length, with cross sectional dimensions of 0.5 inches wide by 0.031 inches thick. It is modeled with forty B31 beam elements in Abaqus, resulting in 123 DOF. It is constructed of steel with a Young's modulus of 29,700 ksi, a shear modulus of 11,600 ksi and a mass density of 7.36×10^{-4} lb-s^2/in^4. The dimensions of the beam come from a benchmark structure used to validate a variety of reduced order modeling techniques [29]. The torsion spring stiffness is denoted as K_t.

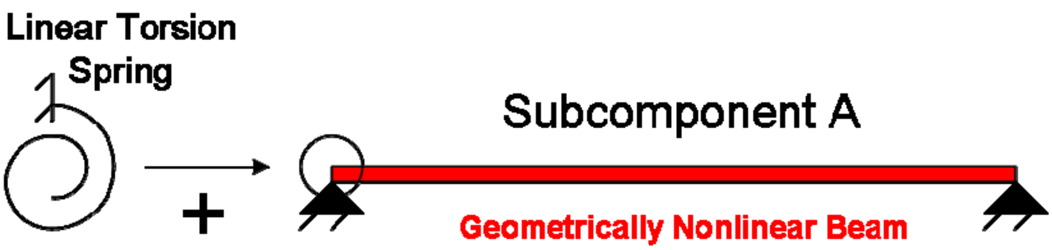

Figure 2: Structural modification case study to understand the effects of an uncertain elastic boundary condition. A torsion spring is coupled to a geometrically nonlinear beam.

The simply supported beam in Fig. 2 represents the unmodified structure to which the torsion spring is attached. A nonlinear ROM is built of this model and used to compute the NNMs of the structure to provide a basis for the quasi-linear modal model used in the structural modification procedure. The NNMs of the modified configuration (e.g. the simply supported beam with the torsion spring attached at the left boundary) are also directly computed by creating an NLROM of a second finite element model that includes the torsion spring. These results are considered the exact solutions for comparison with the structural modification procedure. Here, a torsion spring with a stiffness value of K_t = 25 in-lbf/rad is studied.

3.2 Linear Structural Modification Results

The nonlinear normal modes of a geometrically nonlinear structure converge to the linear normal modes at low energy levels. As the response amplitude becomes small, the nonlinear effects due to large deformations become negligible and the periodic solutions to the conservative equations of motion simply become the linear modes. This is consistent with the assumptions made with linear vibration theory. The first 10 linear normal modes of the finite element models of the simply supported beam and spring-beam assembly are described in Table 1.

Table 1. Exact linear normal modes of unmodified and modified beam

Mode	Simply Supported		Modified with K_t = 25 in-lbf/rad	
	Type	Frequency (Hz)	Type	Frequency (Hz)
1	Bending I	34.85	Bending I	45.08
2	Bending II	139.4	Bending II	153.4
3	Bending III	313.8	Bending III	329.7
4	Bending IV	558.2	Bending IV	575.3
5	Bending V	872.7	Bending V	890.7
6	Bending VI	1258	Bending VI	1273
7	Bending VII	1714	Bending VII	1733
8	Bending VIII	2241	Bending VIII	2260
9	Bending IX	2840	Bending IX	2860
10	Bending X	3511	Bending X	3531

A modal convergence study is initially performed on the linear modes of the simply supported beam when a torsion spring with K_t = 25 in-lbf/rad is coupled to the left end. The modal convergence provides insight into the number of NNM solutions to use as a basis to the nonlinear structural modification procedure. The relative percent error of linear natural frequencies from the modal structural modification are presented in Table 2. The predicted frequencies are compared to the exact solutions in Table 1. As the number of modes in the truncated modal basis is increased, the relative error in the predicted natural frequency decreases. In order for the frequency error to be less than 2% for the linear modes, the first 7 bending modes are required. The largest error comes in the first mode of the modified structure, as the low frequency modes have the greatest shift in frequency due to the modification.

Table 2. Percent error of natural frequency

	Number of Linear Modes in Modal Basis of Simply Supported Beam										
	-	1	2	3	4	5	6	7	8	9	10
Modified Beam Mode Number	1	15.6	7.0	4.6	3.4	2.7	2.3	1.9	1.7	1.5	1.3
	2	-	4.8	2.7	1.9	1.5	1.3	1.1	0.9	0.8	0.7
	3	-	-	2.0	1.3	1.0	0.8	0.7	0.6	0.5	0.4
	4	-	-	-	1.0	0.7	0.5	0.5	0.4	0.3	0.3
	5	-	-	-	-	0.6	0.4	0.3	0.3	0.2	0.2
	6	-	-	-	-	-	0.4	0.3	0.2	0.2	0.2
	7	-	-	-	-	-	-	0.3	0.2	0.2	0.1
	8	-	-	-	-	-	-	-	0.2	0.1	0.1
	9	-	-	-	-	-	-	-	-	0.1	0.1
	10	-	-	-	-	-	-	-	-	-	0.1

3.3 Nonlinear Structural Modification Results

Based on the results of the linear modal convergence study, the first 8 NNMs of the geometrically nonlinear beam were used as a basis for the nonlinear structural modification procedure. Hence, the error in the linear modes and the low energy NNMs will be less than 2%. The frequency-energy dependence of each mode in the NNM basis is presented in Fig. 3. The NNMs of the finite element model are computed from two separate nonlinear reduced order models. To minimize the computational cost, the inherent uncoupling due to the symmetry in the beam was exploited and the even modes (NNMs 2, 4, 6, 8) were computed from an NLROM with linear modes 2, 4, 6 and 8. Similarly with the odd modes, an NLROM with linear modes 1, 3, 5 and 7 are used to compute NNMs 1, 3, 5 and 7.

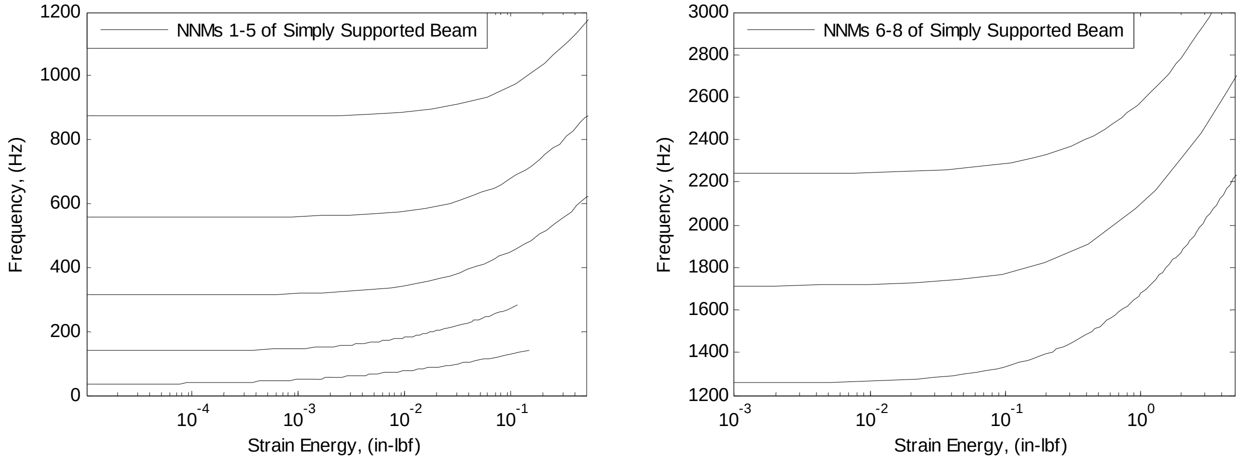

Figure 3: Nonlinear normal modes 1 through 8 for unmodified simply supported beam. This set of modes constitutes the modal basis used in the nonlinear structural modification procedure.

This set of nonlinear modes was used to define a quasi-linear modal model, which was used in the modification procedure to predict the NNMs of the system when a torsion spring with K_t = 25 in-lbf/rad was added to the left end of the beam. The frequency-energy dependence of the predicted NNMs is shown below.

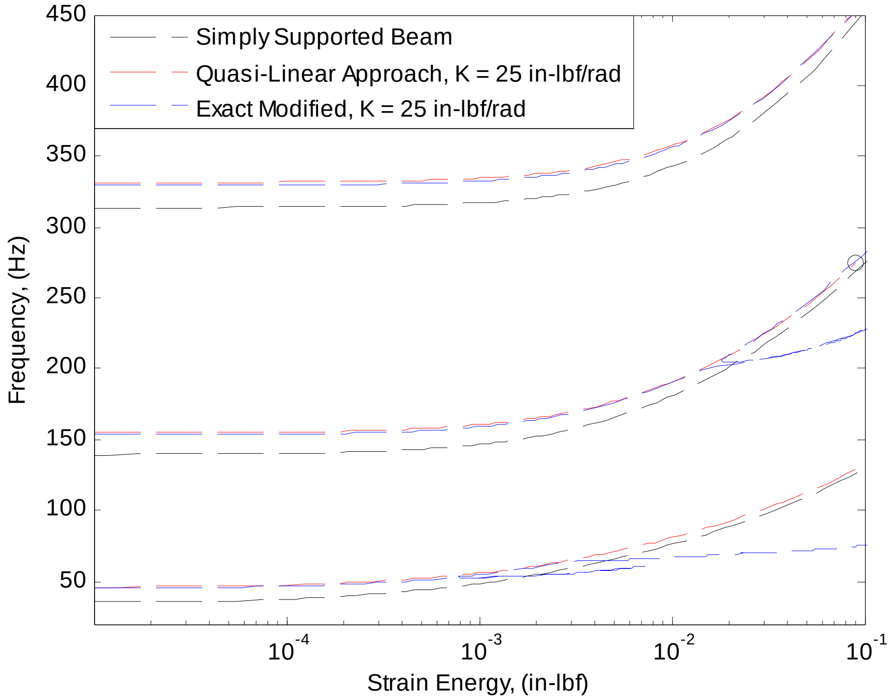

Figure 4: Plot of frequency-energy dependence of first three NNMs of the simply supported beam for the (Black) unmodified NNMs used with QL modal model, (Red) NNMs predicted by the modal structural modification procedure, and (Blue) exact NNMs computed from a finite element model of the beam-spring structure.

The frequency-energy plot shows that the predicted NNMs of the modified structure are in excellent agreement with the exact NNMs. At low energy, where the NNMs converge to the linear normal mode, the error between each of the frequencies is due to modal basis truncation error, as discussed previously. The error in frequencies along the entire solution branch is less than 2% for the first three modes (except where there is an internal resonance). It is expected that the error would decrease further if additional modes were used. By analogy with linear substructuring, one would also expect the error to decrease if the modal basis were improved. For example, we could mass load the interface so that the modes would have nonzero curvature (nonzero moment) at the end. This would constitute a nonlinear extension of the method presented in [30].

To further explore the comparison between the predicted NNMs and the actual ones, we examine the predicted maximum deformation shape of the NNM at a discrete location along the 2nd NNM branch. The maximum deformation shape of the modified structure is compared in Fig. 5 for a solution at the highest energy considered on the 2nd NNM branch (marked with black circle in Fig. 4). The red line represents the response predicted by using the structural modification procedure, and the blue line is the exact solution. There is excellent agreement between these deformation shapes, even when the frequency has shifted over 75% from 154.8 Hz to 274.6 Hz. On the other hand, the deformation shape of this NNM is still dominated by the second linear bending mode shape, with small contributions from other linear modes.

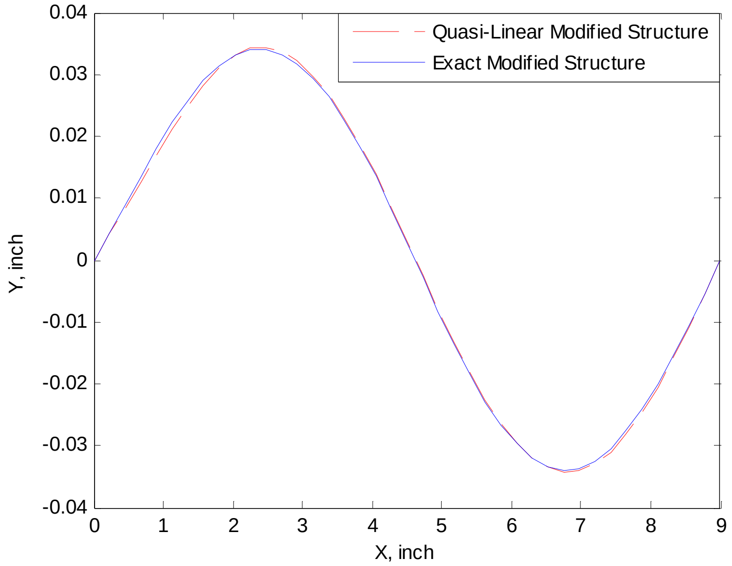

Figure 5: Comparison of the maximum deformation the modified beam for the 2nd NNM at the highest energy considered.

The time histories can also be compared. For example, Fig. 6 shows the transverse displacement at x = 2.25" over one period of the response from the NNM initiated by the initial deformation in Fig. 5. The blue line is the NNM found using the response integrated with the nonlinear reduced order model, and the red line is the quasi-linear response predicted using the structural modification procedure. The QL response is only capable of estimating the fundamental frequency of the NNM, and cannot predict higher harmonics. The exact response is dominated by the fundamental frequency, and the higher order harmonics are relatively small. Therefore, the quasi-linear response is a good approximation to the exact response at this energy level. If the higher harmonics had a greater contribution to the exact response, the QL response would not be as accurate.

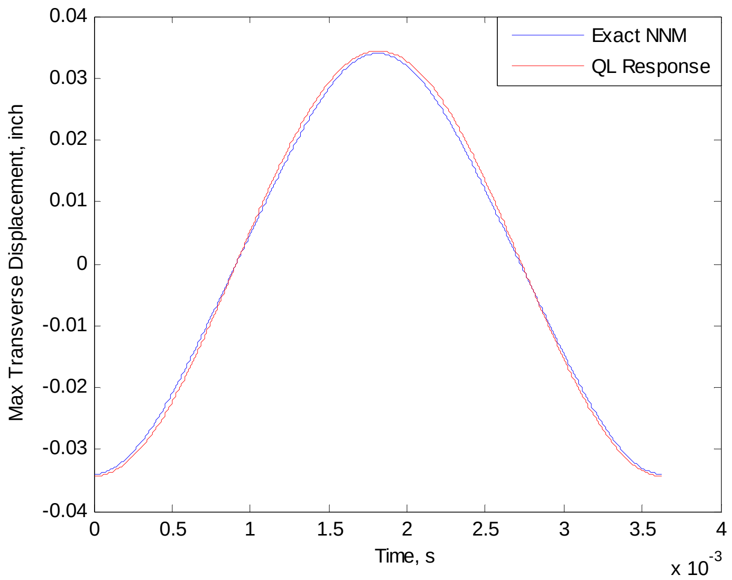

Figure 6: Time history of (red) QL NNM transverse displacement at x = 2.25" compared to the (blue) exact NNM of the modified beam at x = 2.25".

On the other hand, one can see in Fig. 4 that there are energies at which the actual frequency-energy plot contains features that are not captured by the QL model. Since the QL modal model is based on the fundamental harmonic of the response, the modification procedure is only capable of predicting the fundamental harmonic of the assembly. Figures 7 and 8 show expanded views of NNMs 1 and 2, which reveal that there are folds in the frequency-energy curve where there is more than one solution at a discrete energy level. These are known as internal resonances, and occur when two NNMs interact with one another. Typically, these occur when two NNMs have commensurate NNM frequencies at a given energy level.

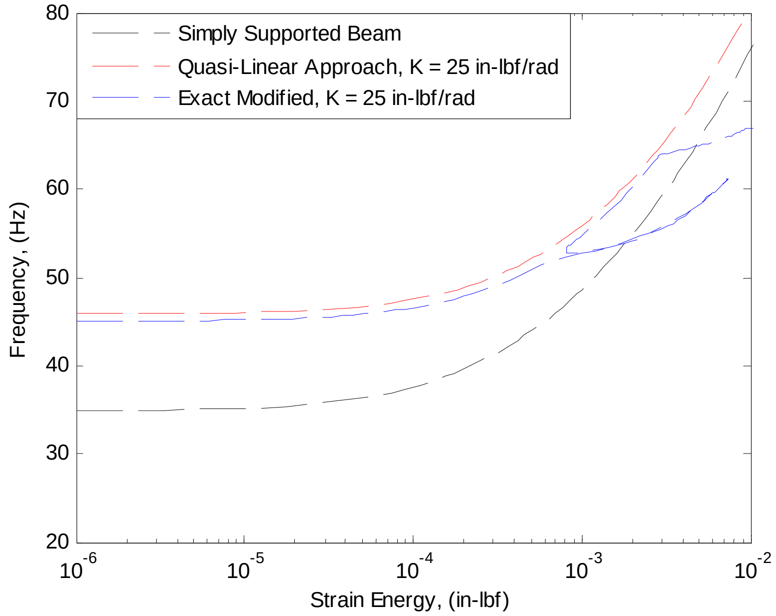

Figure 7: Frequency-energy plot of 1st NNM for (black) unmodified beam (blue) exact modified beam and (red) modified using the QL modification procedure.

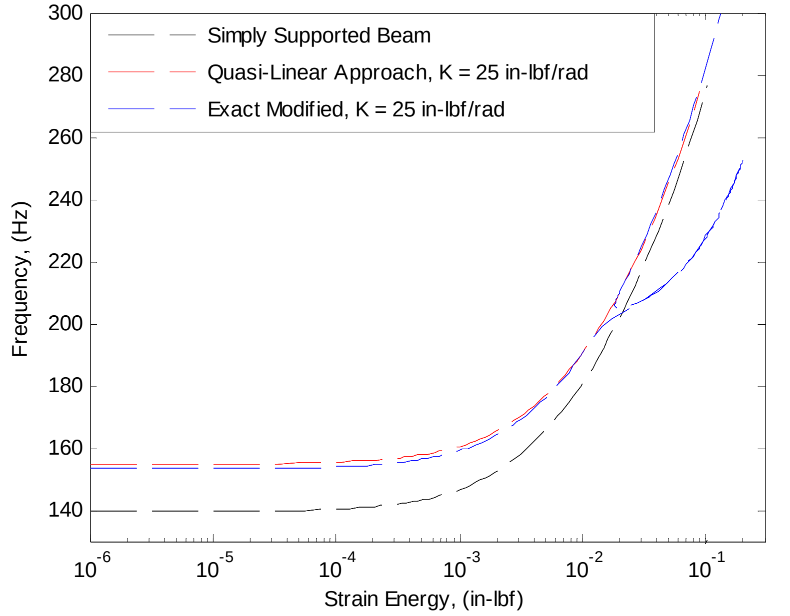

Figure 8: Frequency-energy plot of 2nd NNM for (black) unmodified beam (blue) exact modified beam and (red) modified using the QL modification procedure.

The predicted NNMs from the modification procedure follow the main solution branch quite well when there is no internal resonance present. One can actually predict candidate locations along the NNM branch where an internal resonant condition is possible. For example, the harmonics of the first predicted NNM are overlaid on the frequency-energy plot in Fig. 9. One can see that the 3x harmonic of NNM 1 intersects the predicted frequency-energy curve of NNM 2. One would expect a 3:1 internal resonance in the NNM 1 branch at the energy level where these two solutions intersect. In fact, the first fold in the frequency-energy curve of the exact solution for NMM 1 is a 3:1 internal resonance with the NNM 2.

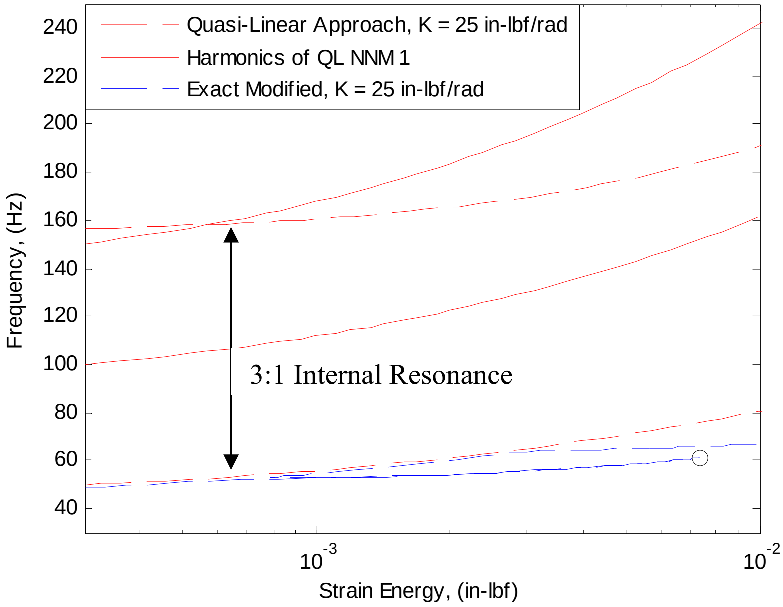

Figure 9: Plot of frequency-energy curves for (Solid Red) harmonics of first predicted NNM, (Dashed Blue) exact NNM 1, and (Dashed Red) modified NNMs.

To further illustrate this internal resonance, the time history of the modal amplitudes are plotted in Fig. 10, taken from the point marked with a black circle in Fig. 9. The response of the modal coordinates q_1 and q_2 from the NLROM of the spring-beam finite element model demonstrates the interaction between the underlying linear modal coordinates. Clearly, the second modal coordinate is dominant and oscillates at 3 times the frequency of the first modal coordinate. The QL model cannot capture this type of response, as discussed previously. However, several investigators [31, 32] have suggested multi-harmonic balance approaches that might be able to capture such features.

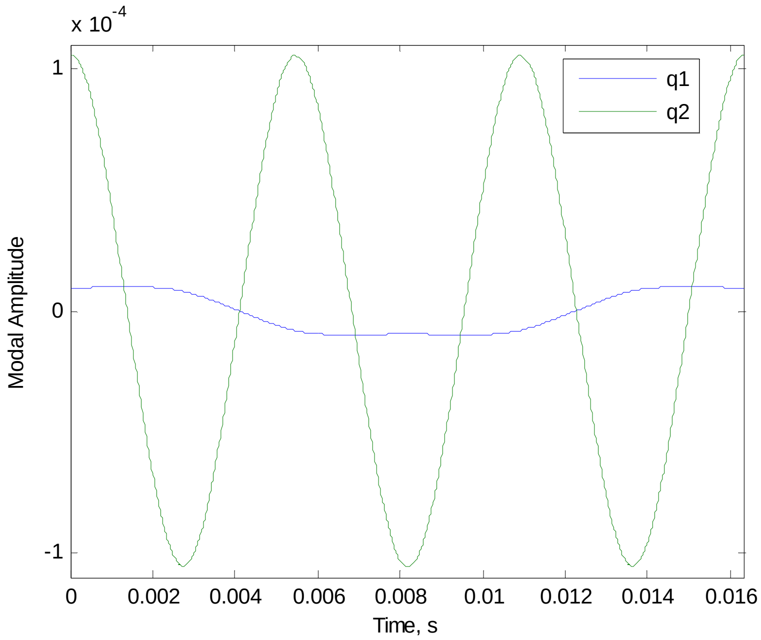

Figure 10: Time history of modal coordinates in NLROM of 3:1 internal resonance.

The main advantage of this approach is the ability to quickly iterate on a nonlinear structural model to predict the nonlinear dynamic behavior due to a modification. At this point the authors are unable to comment on the exact computational savings that would be achieved using this approach, as there were some issues regarding the manner in which the algorithm was implemented. However, our experiences with other similar algorithms suggest that the reduction would be dramatic. This approach avoids the need to recompute the nonlinear reduced order model (using the FEA software) and the need to perform repeated time integrations to predict the NNMs. During early stages of design, when many design configurations are proposed, this approach could provide a means to quickly study the changes to the system.

4. Conclusion

A new structural modification procedure based on the nonlinear normal modes of a structure is presented in this work. The method is validated by applying it to a geometrically nonlinear finite element model modified by the addition of a torsional spring. The results show that the proposed method is capable of accurately predicting the nonlinear modes due to this structural modification, presumably at a reduced computational cost. The discrepancies between the predictions and the truth model at low energy were shown to be due to the truncation error of the basis set used in the QL modal model. Experience with linear systems suggest that the results could be improved by using a richer modal basis for the unmodified structure.

The results also demonstrated that the modification procedure does not capture multi-harmonic responses such as internal resonances. This is due to the fact that the QL modal model only accounts for the fundamental frequency of the response. However, the results did demonstrate that one can use the predicted NNM curves to determine candidate locations of an internal resonance. This method is entirely suitable for design studies where one often wishes to rapidly probe the design space to determine an optimal design. A higher fidelity model could then be used to check the results.

In future work, the method will be extended to a substructuring approach where the NNMs of two finite element model subcomponents will be used to predict the NNMs of the assembly. This approach would allow one to replace computationally expensive subcomponent models with simpler models so the system level behavior can be studied quickly, and it provides the designer with certain insights regarding what types of changes should be made to the subcomponents to produce the desired responses. This type of analysis has been a cornerstone to structural dynamics for over half a decade.

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